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Hidden Measurements, Automorphisms, and Decompositions in Context-Dependent Components

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We investigate in which way the Hilbert space automorphisms can be implemented on the level of Aerts' hidden measurement representations for measurements on physical entities. Inspired by this, we propose a definition for a 'decomposition in context-dependent components' in order to push the property structure of a physical entity on the level of the hidden measurements. We apply this definition within the framework of quantum mechanics and we prove the existence of such a decomposition.

1. INTRODUCTION

Recently, the search for classical representations of quantum structures has known a revival, due to some new results within three different approaches to the foundations of quantum mechanics: in Aerts (1986, 1994) and Coecke (1995, 1996a) within the Brussels–Geneva approach (Jauch, Piron, Aerts, . . .), and in Beltrametti and Bugajski (1996), Singer and Stulpe (1992), and Bush *et al.* (1993) within the statistical (Mackey, Holevo, Prugovecki, Ali, . . .) and/or convex approach (Neumann, Ludwig, Hellwig, Mielnik, . . .). In all these representations, the main key to classical representations consists in considering every quantum measurement as a collection of classical ones equipped with a probability measure. A first 'abstract' step in this direction was made by Gudder (1970). Unfortunately, the search for classical representations still guided by the 'hidden-variable'-idea forced Gudder into an 'enormously large' construction in which most of the structure of the entity got lost (i.e., there is no explicit reference on the level of the hidden variables

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to the lattice structure of the entity). A first explicit model for all quantum entities subjected to measurements with a finite number of outcomes was proposed by Aerts (1986). Aerts considered the quantum state as an as complete a representation of the elements of reality of the entity under study as possible [i.e., he considered '*state*' in the Piron sense of Aerts (1982) and Piron (1976)] and interprets the specific probability structures that appear in quantum mechanics as due to a lack of knowledge on the precise measurement that is actually performed. He expressed this idea in the following way:

- With each real measurement e there corresponds a collection of deterministic measurements e_λ, $\lambda \in \Lambda$, and these deterministic measurements are called 'hidden measurements' in analogy with the 'hidden variables.'
- When a measurement e is performed on an entity in a pure state p, then one of the hidden measurements e_λ takes place. The probability finds its origin in the lack of knowledge about which one of the hidden measurements effectively takes place.

As is shown in Aerts (1986) and Coecke *et al.* (1996), this approach is not in contradiction with the no-go theorems about hidden variables [all inspired by the von Neumann (1955) proof] since the hidden variables in the hidden measurement approach are variables of the measurement environment, which means that they are contextual by definition.[2] It is important to remark that in the hidden measurement approach:

- The state p is not dependent on the parameter λ and the selection of one λ is also independent of the state of the system.

This restriction (and some others) distinguishes between a general contextual hidden variable model and a hidden measurement model. Aerts' approach has been extended beyond the borders of a pure quantum framework (Aerts, 1994) and this has led to a general axiomatics for context dependence and a classification of all possible hidden measurement representations (Coecke, 1995, 1996a, b). This axiomatics will be briefly discussed in Section 2 of this paper. From this classification it follows that in general, there is no feedback from the 'hidden measurement' axiomatics for context dependence to the general axiomatics for physical entities, i.e., the hidden measurement hypothesis has no implications on the level of the lattice description of physical entities (Coecke, 1996a, b). In this paper we proceed more or less

[2] Another model system in which we encounter such an introduction of parameters representative for such a kind of lack of knowledge situation is the Gisin–Piron (1981) model. However, this aspect of the model was not the main topic of their paper. Gisin and Piron mainly wanted to show that it is possible to find a dynamical equation for a 'state transition' during a measurement, i.e., a collapse without a mixture.

along this path of thinking in the sense that we also try to introduce an additional parameter λ in order to decompose a probability measure defined on the property lattice of a physical entity into $\{0,1\}$-valued maps, but now by requiring as general a 'faithfulness' to the property lattice as possible (what we mean by this is explained in Section 3). More precisely, we introduce a decomposition of probability measures assigned to measurements of an entity in a pure state into $\{0,1\}$-valued maps that satisfy certain axioms, and we prove the existence of such a decomposition for the specific case of a Hilbert space quantum framework.

2. ASSUMPTIONS OF THE HIDDEN MEASUREMENT AXIOMATICS FOR CONTEXT DEPENDENCE

In this section we briefly summarize the axiomatics for context dependence introduced in Coecke (1996a, b), i.e., we consider a situation with a lack of knowledge on the interaction of the entity with its measurement context. For the moment, we do not pose any further structural assumptions, except for the following four, which enable us to construct a framework to study these situations:

- The entity is in a (pure) state which is the complete representation of all its properties. The collection of all states is denoted as Σ.
- There exists a set of possible descriptions (= 'relevant' parameters) for the measurement context during the measurement (i.e., a kind of *'states of the measurement context'*), denoted as Λ.
- The result of a measurement is completely determined by the initial conditions: $p \in \Sigma$ and $\lambda \in \Lambda$.
- There exists a statistical description μ_Λ: $\mathcal{B}(\Lambda) \to [0,1]$, with $\mathcal{B}(\Lambda)$ a σ-field of subsets of Λ, for the relative frequency of occurrence of $\lambda \in \Lambda$ in a measurement.

2.1. Mathematical Implementation of Minimal Axioms for Context Dependence

Let us consider a general measurement e. For a fixed $\lambda \in \Lambda$, the measurement process is strictly classical; therefore for every *hidden measurement* e_λ (i.e., the measurement e for a fixed value of λ) there exists a strictly classical observable φ_λ: $\Sigma \to O_e$, where O_e is the set of possible outcomes of the measurement. We can represent the 'unknown but relevant information' for the measurement process as a couple consisting of a set of strictly classical observables $\Phi_\Lambda = \{\varphi_\lambda: \Sigma \to O_e | \lambda \in \Lambda\}$ and a probability measure μ_Λ defined on these observables. For more details on this definition of a hidden measurement representation we refer to Coecke (1996a). For every possible initial state $p \in \Sigma$, a measurement e is characterized by a probability measure $P_{p,e}$: $\mathcal{B}(O_e) \to [0,1]$ ($\mathcal{B}(O_e)$ is a σ-field of subsets of O_e), and thus we can define

$$P_{\Sigma,e}: \Sigma \times \mathcal{B}(O_e) \to [0, 1]: (p, B) \mapsto P_{p,e}(B) \tag{1}$$

Every hidden measurement e_λ corresponds to a (deterministic) $\{0,1\}$-valued probability measure for every $p \in \Sigma$:

$$P_{p,\lambda}(B) = \mathbf{1}_B(\varphi_\lambda(p)) \tag{2}$$

where $\mathbf{1}_B$: $O_e \to \{0, 1\}$ is the indicator of B, i.e., $\forall x \in B$: $\mathbf{1}_B(x) = 1$ and $\forall x \in O_e \backslash B$: $\mathbf{1}_B(x) = 0$. It is easy to see that the existence of a hidden measurement representation corresponds to a decomposition represented in the following diagram:

$$\begin{array}{ccc} \Sigma \times \mathcal{B}(O_e) & \overset{P_{\Sigma,e}}{\to} & [0, 1] \\ \Delta\Lambda\searrow & & \nearrow\mu_\Lambda \\ & \mathcal{B}(\Lambda) & \end{array} \tag{3}$$

where $\Delta\Lambda$: $\Sigma \times \mathcal{B}(O_e) \to \mathcal{B}(\Lambda)$, if it exists, is defined by

$$\Delta\Lambda(p, B) = \{\lambda \in \Lambda | \varphi_\lambda(p) \in B\} \tag{4}$$

and thus we have

$$P_{p,e}(B) = \mu_\Lambda(\Delta\Lambda(p, B)) = \mu_\Lambda(\{\lambda | \mathbf{1}_B(\varphi_\lambda(p)) = 1\}) = \mu_\Lambda(\{\lambda | P_{p,\lambda}(B) = 1\})$$

Thus, for all $p \in \Sigma$ the probability measure $P_{p,e}$ is decomposed into $\{0,1\}$-valued probability measures $P_{p,\lambda}$ according to a weight given by the probability measure μ_Λ. A theorem on the existence of such a hidden measurement representation for finite-dimensional quantum mechanics was given in Aerts (1982). A generalization of this theorem to more general 'finite-dimensional' entities (i.e., with other probability descriptions than quantum probabilities) can be found in Aerts (1994). The general proof for the existence of a hidden measurement representation for general models for physical measurements can be found in Coecke (1995). Coecke (1996a) proved that there always exists a hidden measurement representation with $\Lambda = [0, 1]$ and in Coecke (1996b) we identified and classified all possible hidden measurement representations. In Section 3 of this paper we will try to take the property structure of the entity into account. First we show how the automorphisms of Hilbert space can be implemented within the axiomatics for context dependence.

2.2. Hidden Measurements and Automorphisms of the Hilbert Space Structure

At the end of the previous section, we mentioned some proofs on the existence of at least one hidden measurement representation for general models for physical measurements. In this section we go one step further, in the sense that for the case of entities described in n-dimensional Hilbert spaces, a hidden measurement representation for only one measurement e_0 [which always exists, as is shown in Aerts (1986)] induces one for every other measurement with respect to the automorphisms of the Hilbert space.

Proposition 1. Let $(\Phi_\Lambda, \mu_\Lambda)$ define a hidden measurement representation for a measurement on a quantum entity with states represented in a n-dimensional Hilbert space $\mathcal{H}_n$. The automorphisms of $\mathcal{H}_n$ induce a hidden measurement representation for every quantum measurement on this entity.

Proof. Let $\mathcal{E}$ consist of all quantum measurements on this entity. A measurement $e \in \mathcal{E}$ is represented by a self-adjoint operator H_e: $\mathcal{H}_n \to \mathcal{H}_n$, and due to the spectral theorem, we can represent e by n eigenvectors $p_{e,1}$, . . ., $p_{e,n}$ and n corresponding eigenvalues $o_{e,1}, \ldots, o_{e,n}$. Clearly we can represent the different possible outcomes $o_{e,1}, \ldots, o_{e,n}$ of the measurement by their respective eigenstates $p_{e,1}, \ldots, p_{e,n}$ and thus $O_e \cong \{p_{e,1}, \ldots, p_{e,n}\}$. Consider a given measurement e_0 (with H_0 as self-adjoint operator and $p_{0,1}$, $\ldots$, $p_{0,n}$ as eigenvectors) for which there exist a hidden measurement representation

$$\begin{cases} \Phi_{\Lambda,0} = \{\varphi_{0,\lambda}: \Sigma \to \{p_{0,1}, \ldots, p_{0,n}\} | \lambda \in \Lambda\} \\ \mu_{\Lambda,0}: \mathcal{B}(\Lambda) \to [0, 1] \end{cases} \tag{5}$$

which satisfies (3) and thus characterizes this hidden measurement representation. For every $e \in \mathcal{E}$ let U_e be the unitary transformation defined by $\forall i$: $p_{e,i} = U_e(p_{0,i})$. We can define a representation in the following way:

$$\begin{cases} \Phi_{\Lambda,e} = \{\varphi_{e,\lambda}: \Sigma \to \{p_{e,1}, \ldots, p_{e,n}\}: p \mapsto U_e \circ \varphi_{0,\lambda} \circ U_e^{-1}(p) | \lambda \in \Lambda\} \\ \mu_{\Lambda,e} = \mu_{\Lambda,0} \end{cases} \tag{6}$$

since $U_e \circ H_0 \circ U_e^{-1} = H_e$ if the outcomes of e and e_0 are equal (which is the case since we abstracted them). ■

Thus, the actions 'decomposing according to (3)' and 'transforming the state space under a unitary map' commute. In fact, (6) also imposes an additional assumption on the representation of (5): the representation should be invariant under unitary transformations that preserve e_0 up to a permutation of the outcomes. Let $\mathcal{E}_0$ be all measurements obtained through a permutation of the outcomes of e_0. If we replace the representation of (5) by [$\Phi_{\Lambda,e}$ is defined by (6)]

$$\begin{cases} \cup_{e \in \mathscr{E}_0} \Phi_{\Lambda, e} \\ \mu_{\Lambda \times \mathscr{E}_0}: \mathscr{B}(\Lambda) \times \mathscr{P}(\mathscr{E}_0) \to [0, 1]: (A, B) \mapsto \mu_{\Lambda,0}(A) \cdot \mu_n(B) \end{cases} \tag{7}$$

where $\mathscr{P}(\mathscr{E}_0)$ is the set of all subsets of $\mathscr{E}_0$ and where $\mu_n: \mathscr{P}(\mathscr{E}_0) \to [0,1]$ is the probability measure defined by $\forall_e \in \mathscr{E}_0$: $\mu_n(\{e\}) = 1/n!$ ($n!$ is the number of elements in $\mathscr{E}_0$), this additional condition is clearly fulfilled. We remark that the set Λ is replaced by $\Lambda \times \mathscr{E}_0$.

3. CLASSICAL REPRESENTATIONS OF ENTITIES DESCRIBED BY PROPERTY LATTICES

As remarked in Section 2.1, a hidden measurement representation can be seen as a decomposition of a collection of probability measures in {0, 1}-valued maps (in fact, {0, 1}-valued probability measures). We also already mentioned that there is no feedback from the 'hidden measurement' axiomatics for context dependence to the general axiomatics for physical entities. In this section we will try to introduce an additional parameter λ in order to decompose a probability measure defined on the property lattice of a physical entity into {0,1}-valued maps, but now by pushing a 'faithfulness' as general as possible to the property lattice. A procedure to push the lattice structure into this kind of decomposition can be obtained by going back to the attempts for hidden variable theories in this direction.[3]

Although we formally succeed in this attempt, it is not at all clear if we are able to preserve the spirit of the hidden measurement ideas in the sense formulated by Aerts (see the Introduction). Possibly, a somewhat less mechanistic interpretation will be required to interpret the results of this paper. In order to emphasize this potential spiritual difference between Aerts' hidden measurements e_λ and the {0,1}-valued maps that we obtain in this paper, we feel obliged to express this within the denotation, and to call them 'context dependent components' instead of hidden measurements.[4] The precise significance of the obtained decompositions has already been studied and will appear in forthcoming papers (Coecke and Moore, 1998, n.d.).

[3] A more detailed comparison of the hidden measurement approach and hidden variable approaches can be found in Coecke *et al.* (1996).

[4] In fact, the distinction between 'measurement' and 'entity' becomes less sharp as is the case in the hidden measurement approach. This makes it too difficult to assign the components λ as a variable of the measurement. Therefore we will not denote them by 'e' equipped with an index, since this 'e' refers to a measurement. Instead, we denote a context-dependent component by the variables on which it depends, written between brackets.

3.1. Verification of the Hidden Variable Conditions

As an example we consider the Jauch and Piron (1963) improvement of the von Neumann (1955) theorem. Let $\mathcal{L}$ be the property lattice of a physical entity. Two properties a and b in $\mathcal{L}$ are compatible (denoted $a \leftrightarrow b$) if the sublattice generated by $\{a, a', b, b'\}$ is distributive (a' and b' are the orthocomplements of a and b). A state p (i.e., an atom of the property lattice) is represented by the unique Gleason quantum probability $\omega_p \cdot \mathcal{L} \rightarrow [0, 1]$ given by the square modulus Hilbert space in-product (Gleason, 1957). Jauch and Piron (1963) show that ω_p fulfills

$$\begin{cases} \omega_p(\emptyset) = 0, \qquad \omega_p(I) = 1 \\ a \leftrightarrow b \Rightarrow \omega_p(a) + \omega_p(b) = \omega_p(a \wedge b) + \omega_p(a \vee b) \\ \omega_p(a) = \omega_p(b) = 1 \Rightarrow \omega_p(a \wedge b) = 1 \end{cases} \tag{8}$$

Definition 1. A theory is said to admit hidden variables if we can add extra variables Λ_p to every state such that there exist maps (p, λ): $\mathcal{L} \rightarrow \{0,1\}$ and a probability measure μ_p: $\mathcal{B}(\Lambda_p) \rightarrow [0,1]$ [$\mathcal{B}(\Lambda_p)$ is a σ-field of subsets of Λ_p] such that

$$\forall p \in \Sigma, \forall a \in \mathcal{L}, \exists \Lambda_p: \omega_p(a) = \int_{\Lambda_p} (p, \lambda)(a)\, d\mu_p(\lambda) \tag{9}$$

and such that all (p, λ) fulfill (8).

In the hidden measurement approach we can do the following: we relate to every property a a measurement e_a, i.e., an ideal test of the first kind of the property a [for a definition of 'ideal' and 'of the first kind' we refer to Piron (1976)], which itself corresponds to a collection of 'states of the measurement context' Λ_a and a probability measure μ_a: $\mathcal{B}(\Lambda_a) \rightarrow [0,1]$. This gives

$$\forall p \in \Sigma, \forall a \in \mathcal{L}, \exists \Lambda_a: \omega_p(a) = \int_{\Lambda_a} (p, \lambda)(a)\, d\mu_a(\lambda) \tag{10}$$

Moreover, for the specific case of quantum mechanics it is possible to find hidden measurement representations such that Λ and μ do not depend on the properties [see Aerts (1986) and proposition 1], and thus (10) coincides with (9). As a consequence, it seems that at first sight we have a contradiction since there exist hidden measurement representations for quantum mechanics that do satisfy (9). This conflicts with the no-go theorems. As we will show now, (8) is not fulfilled for the maps (p, λ) in the hidden measurement case. To prove this it suffices to consider a two-dimensional Hilbert space. Since this two-dimensional situation can easily be embedded in a higher dimensional Hilbert space, the same is valid for every dimension.

Proposition 2. If (6) is satisfied for e_p and $e_{p'}$,—respectively, tests for the properties p and p'—and (10) is satisfied, then (8) cannot be satisfied.

Proof. Let U be a unitary transformation that preserves $(1/\sqrt{2})(p + p')$ and exchanges p and p'. If we have a decomposition for p and thus also for the probabilities of e_p, we also have one for the probabilities of $e_p{}'$ through U, and thus also for p'. Due to (6) we have $((1/\sqrt{2})(p + p'), \lambda)(p) = ((1/\sqrt{2})(p + p'), \lambda)(p')$. Thus, the sum $((1/\sqrt{2})(p + p'), \lambda)(p) + ((1/\sqrt{2})(p + p'), \lambda)(p')$ is either 0 or 2, where $((1/\sqrt{2})(p + p'), \lambda)(p \vee p') = 1$ and $((1/\sqrt{2})(p + p'), \lambda)(p \wedge p') = 0$, which means that (8) cannot be satisfied.

3.2. Context-Dependent Components

It follows from the previous section that in our decomposition (p, λ) does not fulfill additivity on mutual orthogonal properties. The assumptions by which we will replace this additivity assumption (which has always been taken for granted as an axiom for physical states) seem acceptable and definitely very natural, although they are incompatible with the additivity. The kind of construction that we obtain in this way will be called a 'decomposition in context-dependent components'. A condition which definitely has to be satisfied in such a decomposition is

$$(p, \lambda)(a) \in \{0, 1\} \tag{11}$$

since we demand a deterministic dependence on the initial conditions p and λ [which corresponds with the dispersion-free requirement in Jauch and Piron (1963) and von Neumann (1955) or the restriction of the range of (p, λ) to $\{0,1\}$ in the previous section]. The origin of the implication relation for the properties [i.e., the partial order relation of the lattice; see, Aerts (1982), Jauch (1968), or Piron (1976)] insinuates order preservation[5]: consider two properties a and b and their respective tests e_a and e_b; $a < b \Leftrightarrow$ an answer 'yes' for e_a implies an answer 'yes' for e_b; as a consequence, it is natural to require the same for the 'hidden tests', and thus, following the analogy with the construction in the previous section, also for the components (p, λ). Thus we have

$$a < b \Rightarrow (p, \lambda)(a) \leq (p, \lambda)(b) \tag{12}$$

Clearly, (12) is the implementation of the property lattice on the level of the decomposition. Let F be the collection of automorphisms of $\mathcal{L}$. In analogy

[5]This can also easily be seen in a state space representation of the property lattice where the properties are represented as a closure structure on the state space and the implication is an inclusion [for more details on this representation see Aerts. (1994) and Valckenborgh (1996)]. We do not present the argument within this state space representation in this paper since this would require the introduction of too many additional formal tools.

with the construction in Section 2.2, we also demand for every decomposition in context-dependent components that it respects the symmetries of $\mathcal{L}$, represented within this automorphism group. Thus, for all $f \in F$

$$\forall a \in \mathcal{L}, \forall p \in \Sigma: (f(p), \lambda)(f(a)) = (p, \lambda)(a) \tag{13}$$

This imposes the following condition on (p, λ):

$$\forall f \in F_p, \forall a \in \mathcal{L}, \forall p \in \Sigma: (p, \lambda)(f(a)) = (p, \lambda)(a) \tag{14}$$

where F_p is the stabilizer of p in F:

$$f_p = \{f \in F \mid f(p) = p\} \tag{15}$$

Definition 2. Let ω_p: $\mathcal{L} \to [0, 1]$ be a measure which is additive on mutual orthogonal states, and that takes the value one in the atom p. We call a decomposition for p in context-dependent components any family $\{(p, \lambda)\}_\lambda$ of $\{0,1\}$-valued measures defined on $\mathcal{L}$ such that

$$\forall a \in \mathcal{L}: \omega_p(a) = \int_\Lambda (p, \lambda)(a) \, d\mu(\lambda) \tag{16}$$

such that (11), (12), and (14) are fulfilled.

Once we have such a decomposition for p we have a decomposition for every q for which there exists an automorphism f such that $f(p) = q;$ for this decomposition induced by f we have the same probability measure μ and $(q, \lambda)(a) = (p, \lambda)(f^{-1}(a))$. We show now that such a decomposition exists for the quantum case.

Theorem 1. Let $\mathcal{L}$ be the property lattice of a quantum entity, i.e., $\mathcal{L}$ is isomorphic with the lattice of all projectors on a Hilbert space $\mathcal{H}$. Then for ω_p: $\mathcal{L} \to [0, 1]$ there exists a decomposition in context-dependent components.

Proof. Define $\Lambda = [0, 1]$ and μ: $\mathcal{B}(\Lambda) \to [0, 1]$ as the Lebesque measure, i.e., $\forall \lambda \in [0, 1]$: $\mu([0, \lambda]) = \lambda$. For all $a \in \mathcal{L}$ and for every state p, define $(p, \lambda)(a) = 1$ if $\omega_p(a) \geq \lambda$ and $(p, \lambda)(a) = 0$ if $\omega_p(a) < \lambda$. Now we have to verify (16). We find

$$\begin{aligned}\int_\Lambda (p, \lambda)(a) \, d\mu(\lambda) &= \mu(\{\lambda \mid (p, \lambda)(a) = 1\}) \\ &= \mu(\{\lambda \mid \omega_p(a) \geq \lambda\}) \\ &= \mu([0, \omega_p(a)]) = \omega_p(a)\end{aligned}$$

By the definition of (p, λ), (11) is fulfilled. We also have order preservation since, due to the additivity on mutual orthogonal states, we have $a < b \Rightarrow \omega_p(a) \leq \omega_p(b) \Rightarrow (p, \lambda)(a) \leq (p, \lambda)(b)$. Since $f \in F_p$ preserves the Hilbert in-product and $f(p) = p$ we have $\omega_p(f(a)) = \omega_p(a)$ and thus $(p, \lambda)(f(a)) = 1 \Leftrightarrow \omega_p(f(a)) \geq \lambda \Leftrightarrow \omega_p(a) \geq \lambda \Leftrightarrow (p, \lambda)(a) = 1$ and analogously, $(p, \lambda)(f(a)) = 0 \Leftrightarrow (p, \lambda)(a) = 0$ which completes the proof. ■

This specific property of the existence of a decomposition in context-dependent components has led to a characterization of a specific kind of map called 'atomically generated maps,' studied by Coecke and Moore (1998). Much more can be said about this kind of map and in particular about the general existence and uniqueness in the case of non-quantum property lattices. Moreover, one can also show that these decompositions in context-dependent components are in fact the key tool in order to construct a physically motivated mathematical framework for imperfect experimental procedures. We are preparing a paper on this aspect (Coecke and Moore (n.d.)).

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REFERENCES

Aerts, D. (1982). *Foundations of Physics,* **12**, 1131.
Aerts, D. (1986). *Journal of Mathematical Physics,* **27**, 202.
Aerts, D. (1994). *Foundations of Physics,* **24**, 1227.
Beltrametti, E., and Bugajski, S. (1996). *Journal of Physics A,* **29**, 247.
Bush, P., Hellwig, K. E., and Stulpe, W. (1993). *International Journal of Theoretical Physics,* **32**, 399.
Coecke, B. (1995). *Foundations of Physics Letters,* **8**, 437.
Coecke, B. (1996a). *Helvetica Physica Acta,* **69**, 442.
Coecke, B. (1996b). *Helvetica Physica Acta,* **69**, 462.
Coecke, B., and Moore, D. J. (1996). Decompositions of probability measures on complete ortholattices in join preserving maps, Preprint, Free University of Brussels and Université de Genève.
Coecke, B., and Moore, D. J. (n.d.). A mathematical framework for imperfect experimental procedures, in preparation.
Coecke, B., D'Hooghe, B., and Valckenborgh, F. (1997). In *New Developments on Fundamental Problems in Quantum Physics*, M. Ferrero and A. van der Merwe, eds., p. 103, Plenum Press, New York.

Gisin, N., and Piron, C. (1981). *Letters in Mathematical Physics,* **5**, 379.
Gleason, A. M. (1957). *Journal of Mathematics and Mechanics,* **6**, 885.
Gudder, S. P. (1970). *Journal of Mathematical Physics,* **11**, 431.
Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.
Jauch, J. M., and Piron, C. (1963). *Helvetica Physica Acta,* **36**, 827.
Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.
Singer, M., and Stulpe, W. (1992). *Journal of Mathematical Physics,* **33**, 13.
Valckenborgh, F. (1996). Closure structures and the theorem of decomposition in classical components, *Tatra Mountains Mathematical Publications,* **10**, 75.
Von Neumann, J. (1955). *The Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.